EULER NUMBERS AND POLYNOMIALS OF HIGHER ORDER

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Abstract. The purpose of this paper is to present a systemic study of some families of higher-order q-Euler numbers and polynomials and we construct q-zeta function of order r which interpolates higher-order q-Euler numbers at negative integer.

§1. Introduction/ Preliminaries

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$ (see [16]). When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1-q|_p < 1$. For a fixed $d \in \mathbb{N}$ with $(p,d) = 1$, $d \equiv 1 \ (mod\ 2)$, we set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N, X^* = \bigcup_{\substack{0<a<dp\\(a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$. The binomial formulae are known as

$$(1-b)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i b^i, \text{ where } \binom{n}{l} = \frac{n(n-1)\cdots(n-l+1)}{l!},$$

and

$$\frac{1}{(1-b)^n} = (1-b)^{-n} = \sum_{l=0}^{\infty} \binom{-n}{l} (-b)^l = \sum_{i=0}^{\infty} \binom{n+i-1}{i} b^i.$$

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Recently, many authors have studied the q-extension in the various area(see [4, 5, 6]). In this paper, we try to consider the theory of q-integrals in the p-adic number field associated with Euler numbers and polynomials closely related to fermionic distribution. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x,y) = \frac{f(x)-f(y)}{x-y}$ have a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$(1) \qquad I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [7, 8, 9, 16]).}$$

Thus, we note that

$$(2) \qquad \lim_{q\to 1} I_q(f) = I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x).$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then we have

$$(3) \qquad I_1(f_n) = (-1)^n I_1(f) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l} f(l).$$

Using formula (3), we can readily derive the Euler polynomials, $E_n(x)$, namely,

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ (see [16-20]).}$$

In the special case $x=0$, the sequence $E_n(0) = E_n$ are called the n-th Euler numbers. In one of an impressive series of papers(see[1, 2, 3, 21, 23]), Barnes developed the so-called multiple zeta and multiple gamma functions. Barnes' multiple zeta function $\zeta_N(s, w|a_1, \cdots, a_N)$ depend on the parameters $a_1, \cdots, a_N$ that will be assumed to be positive. It is defined by the following series:
(4)

$$\zeta_N(s,w|a_1,\cdots,a_N) = \sum_{m_1,\cdots,m_N=0}^{\infty} (w+m_1a_1+\cdots+m_Na_N)^{-s} \text{ for } \Re(s) > N, \Re(w) > 0.$$

From (4), we can easily see that

$$\zeta_{M+1}(s, w+a_{M+1}|a_1,\cdots,a_{N+1}) - \zeta_{M+1}(s,w|a_1,\cdots,a_{N+1}) = -\zeta_M(s,w|a_1,\cdots,a_N),$$

and $\zeta_0(s,w) = w^{-s}$(see [1]). Barnes showed that ζ_N has a meromorphic continuation in s (with simple poles only at $s = 1, 2, \cdots, N$ and defined his multiple gamma function $\Gamma_N(w)$ in terms of the s-derivative at $s = 0$, which may be recalled here

as follows: $\psi_n(w|a_1,\cdots,a_N)=\partial_s\zeta_N(s,w|a_1,\cdots,a_N)|_{s=0}$(see[11]). Barnes' multiple Bernoulli polynomials $B_n(x,r|a_1,\cdots,a_r)$ are defined by

$$(5)\quad \frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}e^{xt}=\sum_{n=0}^{\infty}B_n(x,r|a_1,\cdots,a_r)\frac{t^n}{n!},\ (|t|<\max_{1\le i\le r}\frac{2\pi}{|a_i|}),\ (\text{see [1, 11]}).$$

By (4) and (5), we see that

$$\zeta_N(-m,w|a_1,\cdots,a_N)=\frac{(-1)^N m!}{(N+m)!}B_{N+m}(w,N|a_1,\cdots,a_N),\ (\text{see [1]}),$$

where $w>0$ and m is a positive integer. By using the fermionic p-adic q-integral on $\mathbb{Z}_p$, we consider the Barnes' type multiple q-Euler polynomials and numbers in this paper. The main purpose of this paper is to study a systemic properties of some families of higher-order q-Euler polynomials and numbers. Finally, we construct q-zeta function of order r which interpolates higher-order q-Euler numbers and polynomials at negative integer.

§2. higher-order q-Euler numbers and polynomials

Let $x,w_1,w_2,\cdots,w_r$ be complex numbers with positive real parts. In $\mathbb{C}$, the Barnes type multiple Euler numbers and polynomials are defined by

(6)
$$\frac{2^r}{\prod_{j=1}^r(e^{w_jt}+1)}e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x|w_1,\cdots,w_r)\frac{t^n}{n!},\ \text{for } |t|<max\{\frac{\pi}{|w_i|}|i=1,\cdots,r\},$$

and $E_n^{(r)}(w_1,\cdots,w_r)=E_n^{(r)}(0|w_1,\cdots,w_r)$(see [11, 12, 14]). In this section, we assume that $q\in\mathbb{C}_p$ with $|1-q|_p<1$. We first consider the q-extension of Euler polynomials as follows:

$$(7)\quad \sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}q^ye^{(x+y)t}d\mu_1(y)=2\sum_{m=0}^{\infty}(-q)^me^{(m+x)t}=\frac{2}{qe^t+1}e^{xt}.$$

In the special case $x=0$, $E_{n,q}=E_{n,q}(0)$ are called the q-Euler numbers. By using multivariate p-adic invariant integral on $\mathbb{Z}_p$, we consider the q-Euler polynomials of order $r\in\mathbb{N}$ as follows:

$$(8)\quad \begin{aligned}\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{t^n}{n!}&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x+x_1+\cdots+x_r)t}q^{x_1+\cdots+x_r}d\mu_1(x_1)\cdots d\mu_1(x_r)\\&=\left(\frac{2}{qe^t+1}\right)^re^{xt}=2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-q)^me^{(m+x)t}.\end{aligned}$$

In the special case $x = 0$, the sequence $E^{(r)}_{n,q}(0) = E^{(r)}_{n,q}$ are refereed as the q-extension of the Euler numbers of order r. Let $f \in \mathbb{N}$ with $f \equiv 1$ $(mod\ 2)$. Then we have

$$\begin{aligned} E^{(r)}_{n,q}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{x_1+\cdots+x_r}(x+x_1+\cdots+x_r)^n d\mu_1(x_1)\cdots d\mu_1(x_r) \\ &= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r}(m_1+\cdots+m_r+x)^n. \end{aligned} \tag{9}$$

By (8) and (9), we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned} E^{(r)}_{n,q}(x) &= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r}(m_1+\cdots+m_r+x)^n \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-q)^m (m+x)^n. \end{aligned}$$

Let $F^{(r)}_q(t,x) = \sum_{n=0}^{\infty} E^{(r)}_{n,q}(x)\frac{t^n}{n!}$. Then we have

$$\begin{aligned} F^{(r)}_q(t,x) &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-q)^m e^{(m+x)t} \\ &= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-q)^{m_1+\cdots+m_r} e^{(m_1+\cdots+m_r+x)t}. \end{aligned} \tag{10}$$

Let χ be the Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1$ $(mod\ 2)$. Then the generalized q-Euler polynomials attached to χ are defined by

$$\sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty}(-q)^m \chi(m) e^{(m+x)t}. \tag{11}$$

Thus, we have

$$\begin{aligned} &E_{n,\chi,q}(x) \\ &= \sum_{a=0}^{f-1} \chi(a)(-q)^a \int_{\mathbb{Z}_p} (x+a+fy)^n q^{fy} d\mu_1(y) = f^n \sum_{a=0}^{f-1} \chi(a)(-q)^a E_{n,q^f}(\frac{x+a}{f}). \end{aligned} \tag{12}$$

In the special case $x = 0$, the sequence $E_{n,\chi,q}(0) = E_{n,\chi,q}$ are called the n-th generalized q-Euler numbers attached to χ. From (2) and (3), we can easily derive the following equation.

$$E_{m,\chi,q}(nf) - (-1)^n E_{m,\chi,q} = 2\sum_{l=0}^{nf-1}(-1)^{n-1-l}\chi(l)q^l l^m.$$

Let us define higher-order generalized q-Euler polynomials attached to χ as follows:

$$\begin{aligned}(13)\quad &\int_X \cdots \int_X \left(\prod_{i=1}^{r}\chi(x_i)\right) e^{(x_1+\cdots+x_r+x)t} q^{x_1+\cdots+x_r} d\mu_1(x_1)\cdots d\mu_1(x_r)\\ &= \left(\frac{2\sum_{a=0}^{f-1}(-q)^a\chi(a)e^{at}}{q^f e^{ft}+1}\right) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!},\end{aligned}$$

where $E_{n,\chi,q}^{(r)}(x)$ are called the n-th generalized q-Euler polynomials of order r attached to χ. By (13), we see that

(14)

$$\begin{aligned}&E_{n,\chi,q}^{(r)}(x)\\ &= 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-q^f)^m \sum_{a_1,\cdots,a_r=0}^{f-1}(\prod_{j=1}^{r}\chi(a_j))(-q)^{\sum_{i=1}^{r}a_i}(\sum_{j=1}^{r}a_j+x+mf)^n,\end{aligned}$$

and

$$(15)\quad \sum_{n=0}^{\infty}E_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!} = 2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-q)^{\sum_{j=1}^{r}m_j}\left(\prod_{i=1}^{r}\chi(m_i)\right)e^{(x+\sum_{j=1}^{r}m_j)t}.$$

In the special case $x=0$, the sequence $E_{n,\chi,q}^{(r)}(0)=E_{n,\chi,q}^{(r)}$ are called the n-th generalized q-Euler numbers of order r attached to χ.

By (14) and (15), we obtain the following theorem.

Theorem 2. *Let χ be the Dirichlet's character with conductor $f\in\mathbb{N}$ with $f\equiv 1$ (mod 2). For $n\in\mathbb{Z}_+$, $r\in\mathbb{N}$, we have*

$$\begin{aligned}&E_{n,\chi,q}^{(r)}(x)\\ &= 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-q^f)^m \sum_{a_1,\cdots,a_r=0}^{f-1}(\prod_{j=1}^{r}\chi(a_j))(-q)^{\sum_{i=1}^{r}a_i}(\sum_{j=1}^{r}a_j+x+mf)^n\\ &= 2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-q)^{m_1+\cdots+m_r}\left(\prod_{i=1}^{r}\chi(m_i)\right)(x+m_1+\cdots+m_r)^n.\end{aligned}$$

For $h\in\mathbb{Z}$ and $r\in\mathbb{N}$, we introduce the extended higher-order q-Euler polynomials as follows:

$$(16)\quad E_{n,q}^{(h,r)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{j=1}^{r}(h-j)x_j}(x+x_1+\cdots+x_r)^n d\mu_1(x_1)\cdots d\mu_1(x_r).$$

From (16), we note that
(17)
$$E_{n,q}^{(h,r)}(x)=2^r\sum_{m_1,\cdots,m_r=0}^{\infty}q^{(h-1)m_1+\cdots+(h-r)m_r}(-1)^{m_1+\cdots+m_r}(x+m_1+\cdots+m_r)^n,$$

where $\binom{n}{l}_q=\frac{[n]_q[n-1]_q\cdots[n-l+1]_q}{[l]_q[l-1]_q\cdots[2]_q[1]_q}$ and $[n]_q=\frac{1-q^n}{1-q}$.

Thus, we have

$$E_{n,q}^{(h,r)}(x)=2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-q^{h-r})^m(x+m)^n. \tag{18}$$

Let
$$\begin{aligned}F_q^{(h,r)}(t,x)&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{j=1}^{r}(h-j)x_j}e^{(\sum_{i=1}^{r}x_i+x)t}d\mu_1(x_1)\cdots d\mu_1(x_r)\\&=\sum_{n=0}^{\infty}E_{n,q}^{(h,r)}(x)\frac{t^n}{n!}.\end{aligned}$$

Then we have
(19)
$$\begin{aligned}F_q^{(h,r)}(t,x)&=\frac{2^r}{\prod_{j=1}^{r}(1+e^tq^{h-r+j-1})}e^{xt}=2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-q^{h-r})^me^{(m+x)t}\\&=2^r\sum_{m_1,\cdots,m_r=0}^{\infty}q^{\sum_{j=1}^{r}(h-j)m_j}(-1)^{\sum_{j=1}^{r}m_j}e^{(x+m_1+\cdots+m_r)t}.\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 3. *For $h,\in\mathbb{Z}$, $r\in\mathbb{N}$, and $x\in\mathbb{Q}^+$, we have*

$$\begin{aligned}E_{n,q}^{(h,r)}(x)&=2^r\sum_{m_1,\cdots,m_r=0}^{\infty}q^{(h-1)m_1+\cdots+(h-r)m_r}(-1)^{m_1+\cdots+m_r}(m_1+\cdots+m_r+x)^n\\&=2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-q^{h-r})^m(x+m)^n.\end{aligned}$$

For $f\in\mathbb{N}$ with $f\equiv 1$ $(mod\ 2)$, it is easy to show that the following distribution relation for $E_{n,q}^{(h,r)}(x)$.

$$E_{n,q}^{(h,r)}(x)=f^n\sum_{a_1,\cdots,a_r=0}^{f-1}(-1)^{a_1+\cdots+a_r}q^{\sum_{j=1}^{r}(h-j)a_j}E_{n,q^f}(\frac{x+a_1+\cdots+a_r}{f}).$$

Let us consider Barnes' type higher-order q-Euler polynomials. For $w_1, \cdots, w_r \in \mathbb{Z}_p$, we define the Barnes' type q- Euler polynomials of order r as follow:

$$
\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|w_1,\cdots,w_r)\frac{t^n}{n!} &= \frac{2^r}{\prod_{i=1}^{r}(e^{w_it}q^{w_i}+1)}e^{xt} \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(\sum_{j=1}^{r} w_jx_j+x)t}q^{w_1x_1+\cdots+w_rx_r}d\mu_1(x_1)\cdots d\mu_1(x_r).
\end{aligned} \tag{20}
$$

From (20), we can easily derive the following equation.

(21)

$$
E_{n,q}^{(r)}(x|w_1,\cdots,w_r) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\sum_{i=1}^{r}x_iw_i+x)^n q^{w_1x_1+\cdots+x_rw_r}d\mu_1(x_1)\cdots d\mu_1(x_r).
$$

Thus, we have

$$
\begin{aligned}
&E_{n,q}^{(r)}(x|w_1,\cdots,w_r) \\
&= f^n \sum_{a_1,\cdots,a_r=0}^{f-1}(-1)^{\sum_{i=1}^{r}a_i}q^{\sum_{j=1}^{r}w_ja_j}E_{n,q^f}^{(r)}(\frac{\sum_{j=1}^{r}w_ja_j+x}{f}|w_1,\cdots,w_r),
\end{aligned} \tag{22}
$$

where $f \in \mathbb{N}$ with $f \equiv 1 \ (mod\ 2)$. By (22), we see that

$$
E_{n,q}^{(r)}(x|w_1,\cdots,w_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty}(-q)^{m_1w_1+\cdots+m_rw_r}(x+w_1m_1+\cdots+w_rm_r)^n. \tag{23}
$$

In the special case $x = 0$, the sequence $E_{n,q}^{(r)}(w_1,\cdots,w_r) = E_{n,q}^{(r)}(0|w_1,\cdots,w_r)$ are called the n-th Barnes' type q-Euler numbers of order r.

Let $F_q^{(r)}(t,x|w_1,\cdots,w_r) = \sum_{n=0}^{\infty}E_{n,q}^{(r)}(x|w_1,\cdots,w_r)\frac{t^n}{n!}$. Then we have

$$
F_q^{(r)}(t,x|w_1,\cdots,w_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty}(-q)^{m_1w_1+\cdots+m_rw_r}e^{(x+w_1m_1+\cdots+w_rm_r)t}. \tag{24}
$$

Therefore we obtain the following theorem.

Theorem 4. *For $w_1,\cdots,w_r \in \mathbb{Z}_p$, $r \in \mathbb{N}$, and $x \in \mathbb{Q}^+$, we have*

$$
E_{n,q}^{(r)}(x|w_1,\cdots,w_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty}(-q)^{m_1w_1+\cdots+m_rw_r}(x+m_1w_1+\cdots+m_rw_r)^n.
$$

For $w_1,\cdots,w_r \in \mathbb{Z}_p$, $a_1,\cdots,a_r \in \mathbb{Z}$, we consider another q-extension of Barnes' type q-Euler polynomials of order r as follows:

(25)

$$
\sum_{n=0}^{r}E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)\frac{t^n}{n!} = \frac{2^r}{(q^{a_1}e^{w_1t}+1)(q^{a_2}e^{w_2t}+1)\cdots(q^{a_r}e^{w_rt}+1)}.
$$

Thus, we have

$$E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)$$
$$=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x+\sum_{j=1}^r w_jx_j)t}q^{\sum_{j=1}^r a_jx_j}d\mu_1(x_1)\cdots d\mu_1(x_r). \tag{26}$$

From (25) and (26), we note that

(27)

$$E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)=2^r\sum_{m_1,\cdots,m_r=0}(-1)^{\sum_{j=1}^r m_j}q^{\sum_{i=1}^r a_im_i}(x+\sum_{j=1}^r w_jx_j)^n.$$

Let $F_q^{(r)}(t,x|w_1,\cdots,w_r;a_1,\cdots,a_r)=\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)\frac{t^n}{n!}$. Then, we see that

$$F_q^{(r)}(t,x|w_1,\cdots,w_r;a_1,\cdots,a_r)$$
$$=2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}q^{a_1m_1+\cdots+a_rm_r}e^{(x+w_1m_1+\cdots+w_rm_r)t}. \tag{28}$$

Theorem 5. *For $r\in\mathbb{N}$, $w_1,\cdots,w_r\in\mathbb{Z}_p$, and $a_1,\cdots,a_r\in\mathbb{Z}$, we have*

$$E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots a_r)=2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{\sum_{j=1}^r m_j}q^{\sum_{i=1}^r a_im_i}(x+\sum_{j=1}^r w_jm_j)^n.$$

Let χ be a Dirichlet's character with conductor $f\in\mathbb{N}$ with $f\equiv 1$ $(mod\ 2)$. By using multivariate p-adic invariant integral on X, we now consider the generalized Barnes' type q-Euler polynomials of order r attached to χ as follows:

$$\sum_{n=0}^{\infty}E_{n,\chi,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)\frac{t^n}{n!}$$
$$=\int_X\cdots\int_X e^{(x+w_1x_1+\cdots+w_rx_r)t}\left(\prod_{j=1}^r\chi(x_j)\right)q^{a_1x_1+\cdots+a_rx_r}d\mu_1(x_1)\cdots d\mu_1(x_r).$$

Thus, we have

$$\left(\frac{2\sum_{b_1=0}^{f-1}\chi(b_1)q^{a_1b_1}(-1)^{b_1}e^{w_1b_1t}}{q^{a_1f}e^{w_1ft}+1}\right)\times\cdots\times\left(\frac{\sum_{b_r=0}^{f-1}\chi(b_r)q^{a_rb_r}(-1)^{b_r}e^{w_rb_rt}}{q^{a_rf}e^{w_rft}+1}\right)$$
$$=\sum_{n=0}^{\infty}E_{n,\chi,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)\frac{t^n}{n!}. \tag{29}$$

From (29), we have

$$E_{n,\chi,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)$$
$$=2^r\sum_{m_1,\cdots,m_r=0}^{\infty}\left(\prod_{j=1}^r\chi(m_i)\right)(-1)^{m_1+\cdots+m_r}q^{a_1m_1+\cdots+a_rm_r}(x+\sum_{j=1}^r w_jm_j)^n.$$

Therefore we obtain the following theorem.

Theorem 6. *For $r \in \mathbb{N}$, $w_1, \cdots, w_r \in \mathbb{Z}_p$, and $a_1, \cdots, a_r \in \mathbb{Z}$, we have*

$$E_{n,\chi,q}^{(r)}(x|w_1, \cdots, w_r; a_1, \cdots, a_r) = 2^r \sum_{m_1, \cdots, m_r=0}^{\infty} \left(\prod_{j=1}^{r} \chi(m_i) \right) (-1)^{m_1+\cdots+m_r} q^{a_1 m_1 + \cdots + a_r m_r} (x + \sum_{j=1}^{r} w_j m_j)^n.$$

Let $F_{q,\chi}^{(r)}(t, x|w_1, \cdots, w_r; a_1, \cdots, a_r) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x|w_1, \cdots, w_r; a_1, \cdots, a_r) \frac{t^n}{n!}$. By Theorem 6, we see that

$$F_{q,\chi}^{(r)}(t, x|w_1, \cdots, w_r; a_1, \cdots, a_r) = 2^r \sum_{m_1, \cdots, m_r=0}^{\infty} \left(\prod_{j=1}^{r} \chi(m_i) \right) (-1)^{m_1+\cdots+m_r} q^{a_1 m_1 + \cdots + a_r m_r} e^{(x+\sum_{j=1}^{r} w_j m_j)t}. \tag{30}$$

§3. Higher-order q-zeta functions in $\mathbb{C}$

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$ and the parameters $w_1, \cdots, w_r$ are positive. From (28), we can define the Barnes' type q-Euler polynomials of order r in $\mathbb{C}$ as follows:

$$\begin{aligned} F_q^{(r)}(t, x|w_1, \cdots, w_r) &= \frac{2^r}{\prod_{j=1}^{r}(e^{w_j t} q^{w_j} + 1)} \\ &= 2^r \sum_{m_1, \cdots, m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} q^{w_1 m_1 + \cdots + w_r m_r} e^{(x+w_1 m_1+\cdots+w_r m_r)t} \\ &= \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|w_1, \cdots, w_r) \frac{t^n}{n!}, \text{ for } |t + \ln q| < \max_{1 \le i \le r} \{ \frac{\pi}{|w_i|} \} . \end{aligned} \tag{31}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, we can derive the following Eq.(32) from the Mellin transformation of $F_q^{(r)}(t, x|w_1, \cdots, w_r)$.

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_q^{(r)}(-t, x|w_1, \cdots, w_r) dt \\ &= 2^r \sum_{m_1, \cdots, m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{m_1 w_1 + \cdots + m_r w_r}}{(x + w_1 m_1 + \cdots + w_r m_r)^s}. \end{aligned} \tag{32}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, we define Barnes' type q-zeta function of order r as follows:

$$\zeta_q^{(r)}(s, x|w_1, \cdots, w_r) = 2^r \sum_{m_1, \cdots, m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{m_1 w_1 + \cdots + m_r w_r}}{(x + w_1 m_1 + \cdots + w_r m_r)^s}. \tag{33}$$

Note that $\zeta_q^{(r)}(s, x|w_1, \cdots, w_r)$ is meromorphic function in whole complex s-plane. By using the Mellin transformation and the Cauchy residue theorem, we obtain the following theorem.

Theorem 7. *For $x \in \mathbb{C}$ with $\Re(x) > 0$, $n \in \mathbb{Z}_+$, we have*

$$\zeta_q^{(r)}(-n, x|w_1, \cdots, w_r) = E_{n,q}^{(r)}(x|w_1, \cdots, w_r).$$

Let χ be a Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1$ $(mod\ 2)$. From (30), we can define the generalized Barnes' type q-Euler polynomials of order r attached to χ in $\mathbb{C}$ as follows:

(34)

$$\begin{aligned}
&F_{q,\chi}^{(r)}(t, x|w_1, \cdots, w_r) \\
&= \left(\frac{2\sum_{b_1=0}^{f-1} \chi(b_1) q^{w_1 b_1} (-1)^{b_1} e^{w_1 b_1 t}}{q^{w_1 f} e^{w_1 f t} + 1} \right) \times \cdots \times \left(\frac{\sum_{b_r=0}^{f-1} \chi(b_r) q^{w_r b_r} (-1)^{b_r} e^{w_r b_r t}}{q^{w_r f} e^{w_r f t} + 1} \right) \\
&= 2^r \sum_{m_1, \cdots, m_r = 0}^{\infty} \left(\prod_{j=1}^{r} \chi(m_i) \right) (-1)^{m_1 + \cdots + m_r} q^{w_1 m_1 + \cdots + w_r m_r} e^{(x + \sum_{j=1}^{r} w_j m_j) t} \\
&= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x|w_1, \cdots, w_r) \frac{t^n}{n!}.
\end{aligned}$$

From (34) and Mellin transformation of $F_{q,\chi}^{(r)}(t, x|w_1, \cdots, w_r)$, we can easily derive the following equation (35) .

$$\begin{aligned}
&\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_{q,\chi}^{(r)}(-t, x|w_1, \cdots, w_r) dt \\
&= 2^r \sum_{m_1, \cdots, m_r = 0}^{\infty} \frac{\left(\prod_{j=1}^{r} \chi(m_i) \right) (-1)^{m_1 + \cdots + m_r} q^{m_1 w_1 + \cdots + m_r w_r}}{(x + w_1 m_1 + \cdots + w_r m_r)^s}.
\end{aligned} \tag{35}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, we also define Dirichlet's type Euler q-l-function of order r as follows:

$$\begin{aligned}
&l_q^{(r)}(s, x; \chi|w_1, \cdots, w_r) \\
&= 2^r \sum_{m_1, \cdots, m_r = 0}^{\infty} \frac{\left(\prod_{j=1}^{r} \chi(m_j) \right) (-1)^{m_1 + \cdots + m_r} q^{m_1 w_1 + \cdots + m_r w_r}}{(x + w_1 m_1 + \cdots + w_r m_r)^s}.
\end{aligned} \tag{36}$$

Note that $l_q^{(r)}(s, x; \chi|w_1, \cdots, w_r)$ is meromorphic function in whole complex s-plane. By using (34), (35), (36), and the Cauchy residue theorem, we obtain the following theorem.

Theorem 8. *For $x, s \in \mathbb{C}$ with $\Re(x) > 0$, $n \in \mathbb{Z}_+$, we have*

$$l_q^{(r)}(-n, x; \chi|w_1, \cdots, w_r) = E_{n,\chi,q}^{(r)}(x|w_1, \cdots, w_r).$$

We note that Theorem 8 is r-ple Dirichlet's type q-l-series. Theorem 8 seems to be interesting and worthwhile for doing study in the area of multiple l-function related to the number theory.

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